A CLASS OF EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR A COMPRESSIBLE HEAT-CONDUCTING GAS

PMM Vol. 33, №3, 1969, pp. 582-584
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(Received November 12, 1968)

Along with the familiar exact solutions of the Navier-Stokes equations describing the flow of an incompressible fluid (the solutions of Hamel, Couette, and Poiseuille), there is a fairly broad class of exact solutions corresponding to jet flows of a viscous fluid. The existence of such a solution was first pointed out by Landau [1]. His solution can be interpreted as the efflux of a jet of finite impulse into infinite space filled with a viscous fluid at rest. Iatseev [3] obtained a class of solutions of the Navier-Stokes equations for the flow of an incompressible fluid from which the Landau and Squire solutions follow as special cases. Wu [4] generalized Iatseev's results for the case of an electrically conductive fluid in a magnetic field. Exact solutions for a compressible heat-conducting gas have been obtained recently. These solutions can be viewed as generalizations of Hamel's solution. Williams [5] obtained the exact solution for axisymmetric source-type flow under the assumption that the coefficients of viscosity and heat conduction depend on the temperature as $(T)^{0.5}$. Byrkin [6] obtained an exact solution analogous to Williams' solution for two-dimensional flow with arbitrary dependence of the viscosity and heat conductivity of the gas on temperature.

We shall obtain a class of exact solutions of the Navier-Stokes equations for two-dimensional flows of a viscous heat-conducting gas. The solutions of Byrkin and Williams are special cases of this class. Our solution coincides with that of Iatseev in the case of an incompressible fluid of constant viscosity.

1. The Navier-Stokes equations describing steady flows of a viscous compressible heat-conducting perfect gas are of the form

$$\nabla(\rho \mathbf{v}) = 0$$

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + {}^1\!/_3 \nabla(\mu\nabla\mathbf{v}) + \mu\Delta\mathbf{v} + \nabla(\nabla\mu\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) - \mathbf{v}\Delta\mu \qquad (1.1)$$

$$Ic_p(\mathbf{v}\cdot\nabla)T - (\mathbf{v}\cdot\nabla)p = I\nabla(k\nabla T) - {}^2\!/_3\mu(\nabla\mathbf{v})^2 + \mu\Delta(\mathbf{v})^2 - 2\mu\mathbf{v}\nabla(\nabla\mathbf{v}) +$$

$$+ 2\mu\nabla\mathbf{v}\times(\nabla\times\mathbf{v}) - \mu(\nabla\times\mathbf{v})^2$$

$$p = \rho RT$$

where ∇ is the Hamiltonian. $\Delta = \nabla^2$; $\mathbf{v}, p, \rho, T, \mu, k, R$ are the velocity vector, pressure, density, temperature, coefficient of viscosity, coefficient of heat conduction, and universal gas constant, respectively. We assume that μ and k are related to temperature by an expression of the form $\mu, k \sim (T)^n$, where n is an arbitrary number.

2. Let us consider axisymmetric flows. We shall make use of spherical coordinates (r, θ, φ) and assume that $v_\varphi \equiv 0$, $\partial(\ldots)/\partial\varphi \equiv 0$ in all equations. We shall attempt to find the solution of Eqs. (1.1) in the form

$$\rho v_r = \Phi(\theta) r^{-\beta}, \quad v_r = \varphi(\theta) r^{-\alpha}, \quad \mu = m(\theta) r^{-\gamma}$$

$$\rho v_\theta = \Psi(\theta) r^{-\beta}, \quad v_\theta = \psi(\theta) r^{-\alpha}, \quad k = \varkappa(\theta) r^{-\gamma} \qquad (2.1)$$

$$T = \tau(\theta) r^{-2\alpha}$$

Existence of a solution of the form (2. 1) requires that

$$\gamma = 2\alpha n, \ \beta = 1 + 2\alpha n \tag{2.2}$$

(α and n are arbitrary numbers)

Substituting relations (2. 1) into our original equations (1. 1) written out in spherical coordinates, we obtain the following system of ordinary differential equations:

$$(\beta - 2)\Phi = \Psi' + \Psi \operatorname{ctg}\theta$$

$$m\varphi'' + [m' + m\operatorname{ctg}\theta - \Psi]\varphi' + [\alpha\Phi + {}^4/_3(\alpha + 1)(1 + \alpha + 2\alpha n)m - 4(\alpha + 1)m]\varphi +$$
$$+ [{}^2/_3\ (1 + \alpha + 2\alpha n) - \alpha - 3]m\psi' + [{}^2/_3(1 + \alpha + 2\alpha n)m\operatorname{ctg}\theta - (\alpha + 1)m' - (\alpha +$$
$$+ 3m)\operatorname{ctg}\theta]\,\Psi + (1 + \alpha + 2\alpha n)R\,(\Phi / \varphi)\tau = 0$$

$${}^4/_3 m\psi'' + [{}^4/_3 m' + {}^4/_3 m\operatorname{ctg}\theta - \Psi]\psi' + [(\alpha - 1)\Phi - {}^2/_3 m'\operatorname{ctg}\theta - 2m\operatorname{ctg}^2\theta + {}^2/_3 m/\sin^{-2}\theta +$$
$$+ (\alpha + 1)(\alpha - 2 + 2\alpha n)m]\,\psi + [{}^8/_3 + {}^1/_3(\alpha - 6\alpha n)]m\varphi' + [{}^2/_3 + {}^2/_3\alpha]m'\varphi -$$
$$- R\,(\Phi / \varphi)'\tau - R\,(\Phi / \varphi)\tau' = 0 \tag{2.3}$$

$$I\varkappa\tau'' + I\ [\varkappa' + \varkappa\operatorname{ctg}\theta - c_v\Psi]\tau' + I[4\alpha^2(1 + n)\varkappa - 2\alpha\varkappa + (\Phi / \varphi)'\psi R / I + 2\alpha c_p\,\Phi -$$
$$- (1 + \alpha + 2\alpha n)\Phi R / I]\tau + m(\varphi')^2 + {}^4/_3 m(\psi')^2 - 2(\alpha + 1)m\psi\varphi' + {}^4/_3(\alpha + 1 - \operatorname{ctg}\theta)\times$$
$$\times m\varphi\psi' + {}^4/_3(\alpha + 1)^2 m\varphi^2 + [(\alpha + 1)^2 + {}^4/_3\operatorname{ctg}^2\theta]m\psi^2 + {}^4/_3(\alpha - 1)\ m\varphi\psi\operatorname{ctg}\theta = 0$$

Here and below the primes denote derivatives with respect to θ.

To Eqs. (2. 3) we must add another relation which follows directly from (2 1),

$$\Phi / \varphi = \Psi/\psi \tag{2.4}$$

The resulting system of equations (2. 3), (2. 4) together with the appropriate boundary conditions is a closed system which enables us to find the unknown functions φ, ψ, τ, Φ, Ψ. We interpret the above two-parameter family of exact solutions as the solution of the jet flow problem. We shall distinguish three cases of jet flow: an unbounded jet, a semi-bounded jet, and a bounded jet. In the first case the range of the solution is $0 \leqslant \theta \leqslant \pi$, in the second case it is $\theta_w \leqslant \theta \leqslant \pi$, and in the third case $-\theta_w \leqslant \theta \leqslant \theta_w$, where θ_w is the vertex half-angle of a circular cone. The boundary conditions in the case of a submerged jet can be written, for example, as

$$\theta = 0, \qquad \psi = \varphi' = \Phi' = \tau' = 0 \tag{2.5}$$
$$\theta = \pi, \qquad \psi = \varphi = 0, \qquad \tau = \tau^*$$

where τ^* is the prescribed static temperature of the ambient gas. In solving (2. 1), (2. 3)–(2. 5) numerically we also specify the values of $\tau(0)$, $\varphi(0)$, $\Phi(0)$, choosing them in such a way as to ensure fulfilment of conditions (2. 5) for $\theta = \pi$. The singularities for $\theta = 0$ and $\theta = \pi$ in Eqs. (2. 3), (2. 4) can be avoided by expressing the required functions as series in θ,

$$\psi = A_1\theta + {}^1/_6 A_3\theta^3 + \ldots, \qquad \Psi = C_1\theta + {}^1/_6 C_3\theta^3 + \ldots$$
$$\varphi = \varphi(0) + {}^1/_2 B_2\theta^2 + \ldots, \qquad \Phi = \Phi(0) + {}^1/_2 D_2\theta^2 + \ldots \tag{2.6}$$
$$\tau = \tau(0) + {}^1/_2 E_2\theta^2 + \ldots, \qquad m = m(0) + {}^1/_2 M_2\theta^2 + \ldots$$
$$k = k(0) + {}^1/_2 K_2\theta^2 + \ldots$$

The relations

$$C_1 = (\beta - 2) / 2\Phi(0), \qquad A_1 = (\beta - 2) / 2\varphi(0)$$

together with (2. 6) now enable us to move away from the singular point $\theta = 0$ to within $O(\theta^2)$ and to proceed with our calculations. A similar device can be used to show that boundary conditions (2. 5) for $\theta = \pi$ can be shifted to a θ close to π to within the same error.

The results of Williams [5] are a special case of the above class of solutions. We can obtain Williams' solution by setting $\alpha = 1$, $n = 1/2$ in the solution of (2. 3), (2. 4); then $\beta = 2$, and the only solution which satisfies the conditions at the axis and walls of the nozzle is a source-type relation, i. e. $\Psi = \psi \equiv 0$.

If we set $\alpha = 1$, $n = 0$, $m = \text{const}$, $\Phi = \varphi$, then the solution of (2. 1), (2. 3)–(2. 5) coincides with that of Iatseev [3].

3. Now let us consider two-dimensional flows. Making use of relations (2. 1), we substitute them into the initial Navier-Stokes equations (1. 1) written out in cylindrical coordinates (r, θ, z), where $v_z \equiv 0$, $\partial(\ldots) / \partial z \equiv 0$. This yields relations (2. 2) and the following system of ordinary differential equations:

$$(\beta - 1)\Phi = \Psi'$$

$$m\varphi'' + [m' - \Psi]\varphi' + [\alpha\Phi + {}^2/_3(2\alpha + 1)(1 + \alpha + 2\alpha n)m - 2(\alpha + 1)m]\varphi + $$
$$+ [{}^2/_3(1 + \alpha + 2\alpha n) - (\alpha + 3)]m\psi' + [\Psi - (\alpha + 1)m']\,\psi + (1 + \alpha + 2\alpha n)R(\Phi / \varphi)\tau = 0$$

$${}^4/_3 m\psi'' + [{}^4/_3 m' - \Psi]\psi' + [(\alpha - 1)\,\Phi + (\alpha + 1)(\alpha - 1 + 2\alpha n)m]\psi + [{}^7/_3 - {}^1/_3 \times$$
$$\times (1 + 6n)\alpha]m\varphi' + {}^2/_3\,[2m' + \alpha m]\varphi - R\,(\Phi/\varphi)'\tau - R\,(\Phi/\varphi)\tau' = 0 \qquad (3.1)$$

$$I\varkappa\tau'' + I\,[\varkappa' - c_v\Psi]\tau' + I\,[4\alpha^2(1 + n)\varkappa + 2\alpha\Phi c_p - (1 + \alpha + 2\alpha n)\Phi R / I +$$
$$+ (\Phi / \varphi)'\psi R / I]\tau + m(\varphi')^2 - 2(\alpha + 1)m'\psi\varphi' + {}^4/_3\,(\alpha + 2)m\varphi\psi +$$
$$+ {}^4/_3\,(\alpha^2 + \alpha + 1)\,m\varphi^2 + (\alpha + 1)^2 m\psi^2 = 0$$

$$\Phi / \varphi = \Psi / \psi$$

The solution of (2. 1), (3. 1) together with the appropriate boundary conditions can be regarded as the exact solution of the problem of a two-dimensional (submerged, semi-bounded, or bounded) jet. No solutions of this type exist in the case of an incompressible fluid, except for Hamel's solution and its generalizations, which describe source- or sink-type flow and can be obtained from the solution of (2. 1), (3. 1) by setting

$$\alpha = 0, \qquad n = 0, \qquad m = \text{const}, \quad \Phi = \varphi$$

Numerical solution of (2. 1), (3. 1) is essentially equivalent to numerical solution of (2. 1), (2. 3), (2. 4) in all three of the above cases of jet flow. Examples are given by Byrkin [6], who solved these systems for $\alpha = 0$, arbitrary n, and $\beta = 1$. The solution of the problem of a bounded jet for these parameter values is a source-type solution, i. e. $\Psi = \psi \equiv 0$.

In conclusion we note the obvious connection between the parameters α and β (or α and n) on the one hand, and the integral characteristics of the jet on the other. The condition of finite jet impulse is $\alpha + \beta = 2$ in the axisymmetric case and $\alpha + \beta = 1$ in the two-dimensional case; the condition of finite mass discharge rate is $\beta = 2$ in the axisymmetric case and $\beta = 1$ in the two-dimensional case. It is clear that the latter conditions are fulfilled in source-type flows only.

The approach described in [4] makes it possible, in principle, to generalize our class of exact solutions for the flow of a viscous compressible electrically conductive gas in a magnetic field.

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Translated by A. Y.

STEADY FLOW OF A VISCOELASTIC FLUID IN A CHANNEL WITH PERMEABLE WALLS

PMM Vol. 33, №3, 1969, pp. 585-591

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(Received December 20, 1968)

The flow of a Maxwellian fluid in a plane channel whose boundaries move at given velocities is considered. The problem need not have a continuous solution in the event of mass flow through the channel boundaries. The factors occasioning a discontinuous solution are discussed. The three-constant Oldroyd model is used as an example to analyze the possible discontinuity structure of the initial two-constant model.

1. Let us assume that the flow of a viscoelastic fluid depends on the single coordinate z and that the behavior of the fluid is described by the rheological equations of Oldroyd's "contravariant" model [1]

$$p_{ij} = -p\delta_{ij} + T_{ij}, \qquad T_{ij} + \lambda_1 T_{ij}^{\cdot} = 2\eta e_{ij}$$
$$T_{ij}^{\cdot} = \partial T_{ij} / \partial t + v_k T_{ij,k} - v_{i,k} T_{kj} - v_{j,k} T_{ik} \tag{1.1}$$

Let the velocity vector be of the form $\mathbf{V} = (v_x,\ 0,\ v_0)$, let the tensor T_{ij} have the nonzero components T_{xx}, T_{xz}, T_{zz}, and let the longitudinal pressure gradient and external body forces equal zero. In the steady-flow case which we shall consider the continuity equation implies that $v_0 = \text{const}$; the equations of motion and relations (1.1) yield the following system of equations closed with respect to $v_x, T_{xz}, T_{xx}, T_{zz}$:

$$\rho v_0 \frac{dv_x}{dz} = \frac{dT_{xz}}{dz}, \qquad T_{xz} + \lambda_1 \left(v_0 \frac{dT_{xz}}{dz} - T_{zz} \frac{dv_x}{dz} \right) = \eta \frac{dv_x}{dz} \tag{1.2}$$

$$T_{xx} + \lambda_1 \left(v_0 \frac{dT_{xx}}{dz} - 2T_{xz} \frac{dv_x}{dz} \right) = 0, \qquad T_{zz} + \lambda_1 v_0 \frac{dT_{zz}}{dz} = 0 \tag{1.3}$$

We are required to find the solution of system (1.2), (1.3) in the domain $|z| \leqslant a$ (a plane channel) which satisfies the following boundary conditions.

We are given the longitudinal velocity $v_x(-a) = u_1$ and $v_x(a) = u_2$ (i.e. we are dealing with Couette-type flow). In addition, we are given the stresses T_{xx} and T_{zz} at the line of entry of the stream into the channel. For example, in the case of injection